

WARM UP

$$\textcircled{1} \frac{4C_3 \cdot 4C_2 \cdot 4C_2}{52C_7} = \frac{144}{133784560} = \boxed{1.08 \times 10^{-6}}$$

$$= \boxed{1.08 \times 10^{-4} \%}$$

$$\textcircled{2} 8C_6 (.25)^6 (.75)^2 + 8C_7 (.25)^7 (.75)^1 + 8C_8 (.25)^8 (.75)^0$$

$$= \boxed{0.0642}$$

$$= \boxed{0.42 \%}$$

**FST 10.3 Notes**

Topic: The Binomial Theorem

GOAL

Explain the connections between combinations, Pascal's Triangle, and binomial coefficients and use the binomial theorem to solve certain counting problems in preparation for the next lesson.

**SPUR Objectives**

- A Expand binomials using the Binomial Theorem.
- D Interpret and describe properties of binomial coefficients combinatorially and algebraically.
- K Represent combinations and binomial coefficients by Pascal's Triangle.

**Vocabulary**

- expansion of  $(x+y)^n$
- binomial coefficients

binomial coefficients The coefficients in the expansion of  $(x + y)^n$ ; the combinations  ${}_n C_k$ .

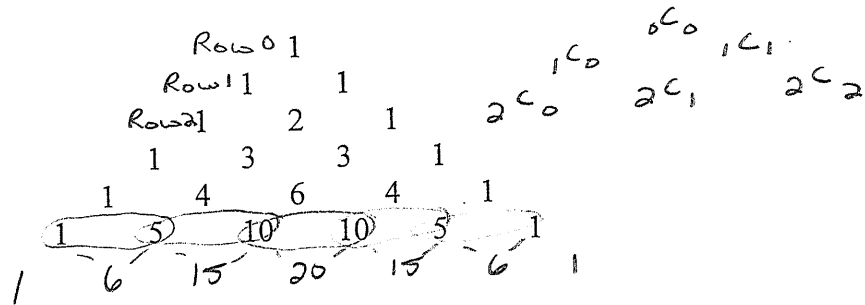
**Mental Math**

Multiply.

- a.  $(4 + 8)(4 + 8) \quad (12)(12) = 144$
- b.  $(x + y)(x + y) \quad x^2 + 2xy + y^2$
- c.  $(40 + 1)(40 + 1) \quad (41)(41) = 1681$

	x	y
x	$x^2$	xy
y	xy	$y^2$

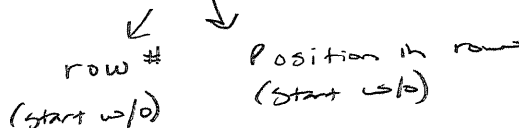
**Review Pascal Triangle**



a) How do we get to the next row?

1's on outside, add 2 #'s above

b) How are the values in the triangle related to  ${}_n C_r$ ?



Review Expanding of binomials

1)  $(x+y)^0 = 1$  (anything to the power of "0" is 1)

2)  $(x+y)^1 = x+y$  (Power of 1, no change)

3)  $(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$

4)  $(x+y)^3 = (x+y)(x+y)(x+y)$   
 $(x^2 + 2xy + y^2)(x+y)$

	$x^2$	$2xy$	$y^2$
$x$	$x^3$	$2x^2y$	$xy^2$
$y$	$x^2y$	$2xy^2$	$y^3$

$x^3 + 3x^2y + 3xy^2 + y^3$   
 1            3            3            1

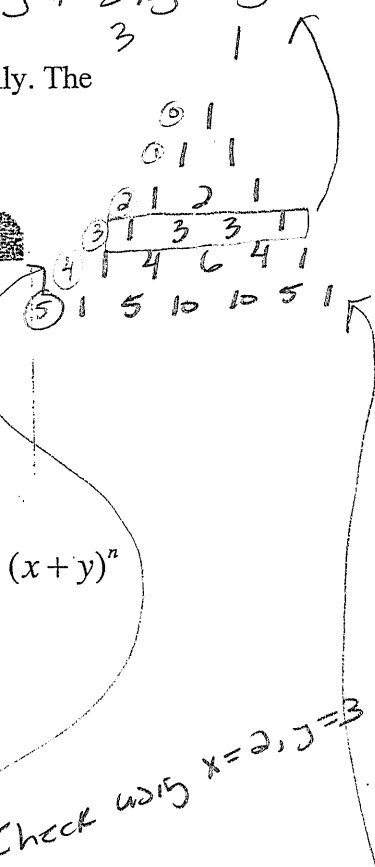
If we try to expand  $(x+y)^4$  it gets more cumbersome and difficult algebraically. The binomial theorem allows us to expand a power of a binomial.

**Binomial Theorem**

For any nonnegative integer  $n$ ,

$$(x+y)^n = {}_n C_0 x^n y^0 + {}_n C_1 x^{n-1} y^1 + {}_n C_2 x^{n-2} y^2 + \dots + {}_n C_k x^{n-k} y^k + \dots + {}_n C_n x^0 y^n$$

$$= \sum_{k=0}^n {}_n C_k x^{n-k} y^k$$



\*\*Each row of Pascal's triangle gives the coefficients for the expansion of  $(x+y)^n$  for positive integers  $n$ .

For example,  $(x+y)^4 = {}_4 C_0 x^4 + {}_4 C_1 x^3 y + {}_4 C_2 x^2 y^2 + {}_4 C_3 x y^3 + {}_4 C_4 y^4$   
 $= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4$   
 1    4    6    4    1

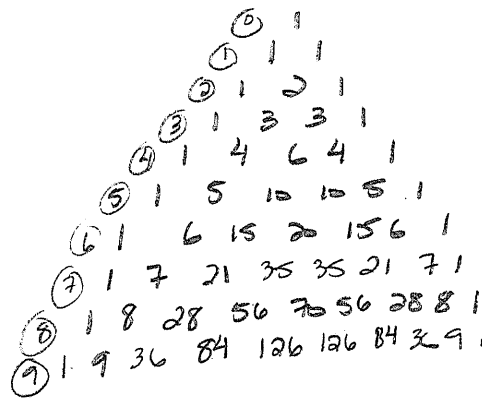
Using the Binomial Theorem expand the following expressions.

Expand  $(x+y)^5$

$$= {}_5 C_0 x^5 y^0 + {}_5 C_1 x^4 y^1 + {}_5 C_2 x^3 y^2 + {}_5 C_3 x^2 y^3 + {}_5 C_4 x^1 y^4 + {}_5 C_5 x^0 y^5$$

$$= 1x^5 y^0 + 5x^4 y^1 + 10x^3 y^2 + 10x^2 y^3 + 5x y^4 + 1x^0 y^5$$

$$= 3125 \checkmark \quad (2+3)^5 = 3125 \checkmark$$



**Example 1**

Find the power of y and the coefficient of the  $x^5$  term in  $(x+y)^9$ .

$$9C4 x^5 y^4$$

$$126 x^5 y^4$$

**Example 2**

Expand  $(3+4y)^5$  and check by letting  $y=1$ .

$$\binom{5}{0} (3)^5 (-4y)^0 + \binom{5}{1} (3)^4 (-4y)^1 + \binom{5}{2} (3)^3 (-4y)^2 +$$

$$\binom{5}{3} (3)^2 (-4y)^3 + \binom{5}{4} (3)^1 (-4y)^4 + \binom{5}{5} (3)^0 (-4y)^5$$

$$= 243y^0 + -1620y^1 + 4320y^2 + -5760y^3 + 3840y^4 + -1024y^5$$

If  $y=1 \Rightarrow 243(1)^0 - 1620(1)^1 + 4320(1)^2 - 5760(1)^3 + 3840(1)^4 - 1024(1)^5 = \boxed{-1}$

$(3-4(1))^5 = \boxed{-1}$

$$\binom{5}{0} H^5 T^0 + \binom{5}{1} H^4 T^1 + \binom{5}{2} H^3 T^2 + \binom{5}{3} H^2 T^3$$

$$+ \binom{5}{4} H^1 T^4 + \binom{5}{5} H^0 T^5$$

**Example 3**

a) A coin is flipped five times. How many of the possible arrangements of heads and tails have at least two heads?

2 or more heads

$(H+T)^5$   
 $2^5 = 32$  total outcomes

Number of Heads	Number of Tails	Sequence Type	Arrangements
2	3	HH TTT	$5C2$ 10
3	2	HHH TT	$5C3$ 10
4	1	HHHHT	$5C4$ 5
5	0	HHHHH	$5C5$ 1
Total			26 ways

Prob:  $\frac{26}{32} = 0.8125$

**Example 3**

b) Find all terms in  $(H + T)^5$  in which the power of H is at least two. *2 or more H's*

Power of H	Power of T	Product Type	Coefficients
2	3	$H^2 T^3$	$5C_3 = 10$
3	2	$H^3 T^2$	$5C_2 = 10$
4	1	$H^4 T^1$	$5C_1 = 5$
5	0	$H^5 T^0$	$5C_0 = 1$

$$\begin{aligned}
 & \overbrace{5C_0}^1 H^5 T^0 + \overbrace{5C_1}^5 H^4 T^1 + \overbrace{5C_2}^{10} H^3 T^2 + \overbrace{5C_3}^{10} H^2 T^3 \\
 & + 5C_4 H^1 T^4 + 5C_5 H^0 T^5
 \end{aligned}$$