WARM UP

① $\frac{{}_4C_3 \cdot {}_4C_2 \cdot {}_4C_2}{{}_{52}C_7} = \frac{144}{133784560} = \boxed{1.08\times 10^{-6} = 1.08\times 10^{-4}\%}$

② ${}_8C_6(.25)^6(.75)^2 + {}_8C_7(.25)^7(.75)^1 + {}_8C_8(.25)^8(.75)^0 = \boxed{0.0042 = 0.42\%}$

FST 10.3 Notes

Topic: The Binomial Theorem

GOAL

Explain the connections between combinations, Pascal's Triangle, and binomial coefficients and use the binomial theorem to solve certain counting problems in preparation for the next lesson.

SPUR Objectives

A Expand binomials using the Binomial Theorem.

D Interpret and describe properties of binomial coefficients combinatorially and algebraically.

K Represent combinations and binomial coefficients by Pascal's Triangle.

Vocabulary

expansion of $(x+y)^n$

binomial coefficients

> binomial coefficients The coefficients in the expansion of $(x + y)^n$; the combinations ${}_nC_k$.

Mental Math

Multiply.

a. $(4 + 8)(4 + 8)$ $(12)(12) = 144$

b. $(x + y)(x + y)$ $x^2 + 2xy + y^2$

c. $(40 + 1)(40 + 1)$ $(41)(41) = 1681$

	x	y
x	x^2	xy
y	xy	y^2

Review Pascal Triangle

Row 0 1

Row 1 1 1

Row 2 1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

${}_0C_0$

${}_1C_0$ ${}_1C_1$

${}_2C_0$ ${}_2C_1$ ${}_2C_2$

a) How do we get to the next row?

1's on outside, add 2 #'s above

b) How are the values in the triangle related to ${}_nC_r$?

n: row # (start w/0)

r: Position in row (start w/0)

Review Expanding of binomials

1) $(x+y)^0$ $=1$ (anything to the power of "0" is 1)

2) $(x+y)^1$ $= x+y$ (Power of 1, no change)

3) $(x+y)^2$ $= (x+y)(x+y) = x^2+xy+xy+y^2 = x^2+2xy+y^2$

4) $(x+y)^3$ $= \underbrace{(x+y)(x+y)}(x+y)$

$(x^2+2xy+y^2)(x+y)$

	x^2	$2xy$	y^2
x	x^3	$2x^2y$	xy^2
y	x^2y	$2xy^2$	y^3

$x^3 + 3x^2y + 3xy^2 + y^3$

1 3 3 1

If we try to expand $(x+y)^4$ it gets more cumbersome and difficult algebraically. The binomial theorem allows us to expand a power of a binomial.

0 1

1 1 1

2 1 2 1

3 $\boxed{1\ 3\ 3\ 1}$

4 1 4 6 4 1

5 1 5 10 10 5 1

Binomial Theorem

For any nonnegative integer n,

$$(x+y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_kx^{n-k}y^k + \cdots + {}_nC_nx^0y^n$$

$$= \sum_{k=0}^{n} {}_nC_kx^{n-k}y^k.$$

****Each row of Pascal's triangle gives the coefficients for the expansion of $(x+y)^n$ for positive integers n.**

For example, $(x+y)^4 = {}_4C_0x^4 + {}_4C_1x^3y + {}_4C_2x^2y^2 + {}_4C_3xy^3 + {}_4C_4y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$

1 4 6 4 1

* Check w/ $x=2, y=3$

Using the Binomial Theorem expand the following expressions.

Expand $(x+y)^5$ $\quad \underbrace{{}_5C_0}_{1} x^5y^0 + \underbrace{{}_5C_1}_{5} x^4y^1 + \underbrace{{}_5C_2}_{10} x^3y^2 + \underbrace{{}_5C_3}_{10} x^2y^3$

$+ \underbrace{{}_5C_4}_{5} x^1y^4 + \underbrace{{}_5C_5}_{1} x^0y^5$

$= 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1x^0y^5$

$= 3125$ ✓

$(2+3)^5 = 3125$ ✓

(0) 1
(1) 1 1
(2) 1 2 1
(3) 1 3 3 1
(4) 1 4 6 4 1
(5) 1 5 10 10 5 1
(6) 1 6 15 20 15 6 1
(7) 1 7 21 35 35 21 7 1
(8) 1 8 28 56 70 56 28 8 1
(9) 1 9 36 84 126 126 84 36 9 1

Example 1

Find the power of y and the coefficient of the x^5 term in $(x+y)^9$.

4 126

$_9C_4\, x^5 y^4$

$126\, x^5 y^4$

Example 2

Expand $(3-4y)^5$ and check by letting y = 1.

$\overbrace{_5C_0}^{1} (3)^5(-4y)^0 + \overbrace{_5C_1}^{5} (3)^4(-4y)^1 + \overbrace{_5C_2}^{10} (3)^3(-4y)^2 +$
$\overbrace{_5C_3}^{10} (3)^2(-4y)^3 + \overbrace{_5C_4}^{5} (3)^1(-4y)^4 + \overbrace{_5C_5}^{1} (3)^0(-4y)^5$

$= 243y^0 + -1620y^1 + 4320y^2 + -5760y^3 + 3840y^4 + -1024y^5$

If $y=1 \Rightarrow 243(1)^0 - 1620(1)^1 + 4320(1)^2 - 5760(1)^3 + 3840(1)^4 - 1024(1)^5 = \boxed{-1}$ ✓

$(3-4(1))^5 = \boxed{-1}$ ✓

$\overbrace{_5C_0}^{1} H^5T^0 + \overbrace{_5C_1}^{5} H^4T^1 + \overbrace{_5C_2}^{10} H^3T^2 + \overbrace{_5C_3}^{10} H^2T^3$
$+ \, _5C_4 H^1T^4 + \, _5C_5 H^0T^5$

Example 3

$(H+T)^5$

a) A coin is flipped five times. How many of the possible arrangements of heads and tails have at least two heads? 2 or more heads

$2^5 = 32$ total outcomes

Number of Heads	Number of Tails	Sequence Type	Arrangements
2	3	HHTTT	$_5C_2$ 10
3	2	HHHTT	$_5C_3$ 10
4	1	HHHHT	$_5C_4$ 5
5	0	HHHHH	$_5C_5$ 1
		Total	26 ways

Prob: $\frac{26}{32} = 0.8125$

Example 3

b) Find all terms in $(H+T)^5$ in which the power of H is at least two. 2 or more H's

Power of H	Power of T	Product Type	Coefficients
2	3	H^2T^3	${}_5C_3 = 10$
3	2	H^3T^2	${}_5C_2 = 10$
4	1	H^4T^1	${}_5C_1 = 5$
5	0	H^5T^0	${}_5C_0 = 1$

$$\underbrace{\overbrace{{}_5C_0}^{1} H^5T^0} + \underbrace{\overbrace{{}_5C_1}^{5} H^4T^1} + \underbrace{\overbrace{{}_5C_2}^{10} H^3T^2} + \underbrace{\overbrace{{}_5C_3}^{10} H^2T^3}$$

$$+ {}_5C_4 H^1T^4 + {}_5C_5 H^0T^5$$