FST 3-7 Notes
Topic: Composition of Functions
GOAL:
Formalize the concept of composition of functions by defining composition and introducing the $\circ$ symbol.

SPUR Objectives

A Find equations for and values of composites of functions.

F Identify properties of composites of functions.

Vocabulary

composite Putting functions together to make one function
function composition

Definition of Composite Function

Suppose f and g are functions. The **composite** of g with f, written $g \circ f$, is the function defined by

$$(g \circ f)(x) = g(f(x)).$$

The domain of $g \circ f$ is the set of values of x in the domain of f for which $f(x)$ is in the domain of g.

* Composition of Functions is not commutative (order matters!)

Example 1: Let $f(x) = x^2$ *and* $g(x) = \dfrac{1}{3x+1}$. Evaluate.

a) $f(g(4))$

$g(4) = \dfrac{1}{3(4)+1}$

$= \dfrac{1}{13}$

$f\left(\frac{1}{13}\right) = \left(\frac{1}{13}\right)^2$

$f(g(4)) = \boxed{\dfrac{1}{169}}$

b) $g(f(4))$

$f(4) = (4)^2$

$= 16$

$g(16) = \dfrac{1}{3(16)+1}$

$g(f(4)) = \boxed{\dfrac{1}{49}}$

c) $(f \circ g)(4) = f(g(4))$

$f\left(\dfrac{1}{3(4)+1}\right)$

$f\left(\frac{1}{13}\right) = \left(\frac{1}{13}\right)^2$

$= \boxed{\dfrac{1}{169}}$

<u>On your own:</u>

Let $f(x) = 3x^2 - 3x$ *and* $g(x) = x + 7$. Evaluate:

a) $(f \circ g)(3) = f(g(3))$

$g(3) = 3 + 7 = 10$

$f(10) = 3(10)^2 - 3(10)$

$= 3(100) - 30$

$= 300 - 30$

$(f \circ g)(3) = \boxed{270}$

b) $g(f(3))$

$f(3) = 3(3)^2 - 3(3)$

$= 27 - 9 = 18$

$g(18) = 18 + 7 = 25$

$g(f(3)) = \boxed{25}$

<u>Example 2</u>: Let $f(x) = x^2$ *and* $g(x) = \dfrac{1}{3x+1}$.

a) Derive a formula for $(f \circ g)(x) = f(g(x))$

$f\left(\frac{1}{3x+1}\right) = \left(\frac{1}{3x+1}\right)^2 = \frac{1^2}{(3x+1)^2} = \frac{1}{(3x+1)(3x+1)} =$

$= \boxed{\dfrac{1}{9x^2+6x+1}}$

b) Give a simplified formula for $(g \circ f)(x) = g(f(x))$

$g(x^2) = \dfrac{1}{3(x^2)+1} = \boxed{\dfrac{1}{3x^2+1}}$

c) Verify that $f \circ g \neq g \circ f$ by graphing.

$\dfrac{1}{9x^2+6x+1} \neq \dfrac{1}{3x^2+1}$

<u>Example 3</u>: Let $f(x) = x^2$ *and* $g(x) = \dfrac{1}{3x+1}$. Find the domain of $f \circ g$.

To be in the domain of $f \circ g$, a number x must be in the domain of g, and the corresponding *g(x)* value must be in the domain of *f.*

Step 1: Find the domain of the "inside" (input) function. If there are any restrictions on the domain, **keep them.**

Step 2: Construct the composite function. Find the domain of this new function. If there are any restrictions on this domain, **add them to the restrictions from Step 1**. If there is an overlap, use the more restrictive domain.

Find the domain of $f \circ g$. $f(g(x))$

① Find domain of inside fxn, $g(x)$.

$g(x) = \dfrac{1}{3x+1}$

$3x+1 \neq 0$
$-1 \quad -1$

$\dfrac{3x}{3} \neq \dfrac{-1}{3}$

$x \neq \dfrac{-1}{3}$

$\boxed{\text{Domain: } \{x \mid x \neq -\tfrac{1}{3}\}}$

② Find $f(g(x))$

$f\left(\dfrac{1}{3x+1}\right) = \left(\dfrac{1}{3x+1}\right)^2 = \dfrac{1}{(3x+1)^2} = \dfrac{1}{(3x+1)(3x+1)} = \dfrac{1}{9x^2+6x+1}$

$9x^2+6x+1 \neq 0$

$(3x+1)(3x+1) \neq 0$

$3x+1 \neq 0$
$-1 \quad -1$

$\dfrac{3x}{3} \neq \dfrac{-1}{3}$

$x \neq -\dfrac{1}{3}$

$\boxed{\text{D: } \{x \mid x \neq -\tfrac{1}{3}\}}$

Example 4: Find $f \circ g$ *and* $g \circ f$ and the domain of each.

$$f(x) = \frac{3x}{x-1} \qquad g(x) = \frac{2}{x}$$

Find $f \circ g$. State the domain. $f(g(x))$

$$f\left(\frac{2}{x}\right) = \frac{3\left(\frac{2}{x}\right)}{\frac{2}{x} - 1} = \frac{\frac{6}{x}}{\frac{2}{x} - \frac{x}{x}} = \frac{\frac{6}{x}}{\frac{2-x}{x}} = \frac{6}{x} \cdot \frac{x}{2-x} = \boxed{\frac{6}{2-x}}$$

$$2 - x \neq 0$$
$$+x \quad +x$$
$$x \neq 2$$

$$g(x) = \frac{2}{x}$$

$$x \neq 0$$

$$\boxed{D: \{x \mid x \neq 0,\ x \neq 2\}}$$

Find $g \circ f$. State the domain. $g(f(x))$

$$g\left(\frac{3x}{x-1}\right) = \frac{2}{\frac{3x}{x-1}} = \frac{2}{1} \cdot \frac{x-1}{3x} = \boxed{\frac{2x-2}{3x}}$$

$$\frac{3x}{3} \neq \frac{0}{3}$$
$$x \neq 0$$

$$f(x) = \frac{3x}{x-1}$$

$$x - 1 \neq 0$$
$$+1 \quad +1$$
$$x \neq 1$$

$$\boxed{D: \{x \mid x \neq 1,\ x \neq 0\}}$$

Example 5. Let $S(x,y)=\left(x,\ \frac{y}{3}\right)$ *and* $T(x,y)=(x-1,\ y-2)$

a) Describe S and T in words.

S: Vertical Shrink by $\frac{1}{3}$

T: Left 1, Down 2

b) Write a simplified formula for the composite $(T\circ S)(x,\ y)$ and describe it in words.

$(T\circ S)(x,y) \rightarrow T(S(x,y))$

$T\left(x, \frac{y}{3}\right)$

$\left(x-1, \frac{y}{3}-2\right)$

- Vertical Shrink by $\frac{1}{3}$, Followed by
- Down 2
- Left 1

c) Write a simplified formula for the composite $(S\circ T)(x,\ y)$ and describe it in words.

$(S\circ T)(x,y) \rightarrow S(x-1, y-2)$

$S\left(x-1, \frac{y-2}{3}\right)$

- Left 1
- Down 2
- Followed by Vertical Shrink by 1/3