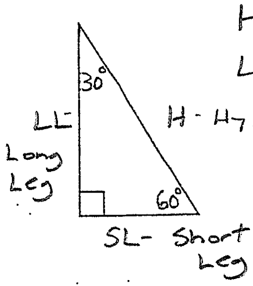
Section 2.5 - Exact Value for Special Angles

Explore/Discuss 1, p. 93

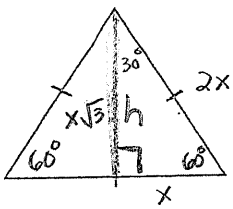
*We can find exact values using our Special Right Triangles

30-60-90



$H = 2 \cdot SL$

$LL = \sqrt{3} \cdot SL$



$x^2 + h^2 = (2x)^2$

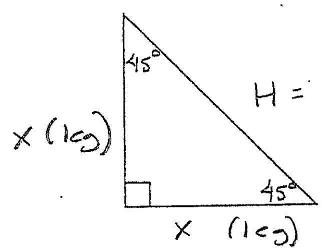
$-x^2 \qquad -x^2$

$h^2 = 4x^2 - x^2$

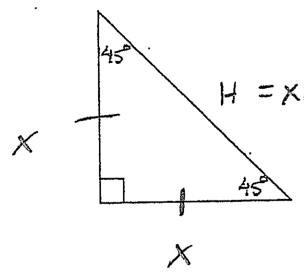
$\sqrt{h^2} = \sqrt{3x^2}$

$\boxed{h = x\sqrt{3}}$

45-45-90



$H = \sqrt{2} \cdot \text{Leg}$



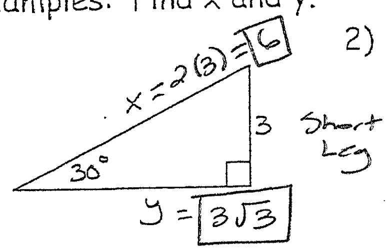
$x^2 + x^2 = H^2$

$H = x\sqrt{2} \quad \sqrt{2x^2} = \sqrt{H^2}$

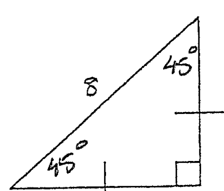
$\boxed{H = x\sqrt{2}}$

Examples: Find x and y.

1)



2)



$x = \boxed{4\sqrt{2}}$

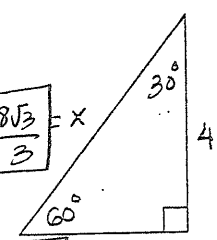
$y = \boxed{4\sqrt{2}}$

$\frac{8}{\sqrt{2}} = \frac{x\sqrt{2}}{\sqrt{2}}$

$x = \frac{8}{\sqrt{2} \cdot \sqrt{2}} \cdot \sqrt{2}$

$= \frac{8\sqrt{2}}{2} = \boxed{4\sqrt{2}}$

3)



$\boxed{\frac{8\sqrt{3}}{3}} = x$

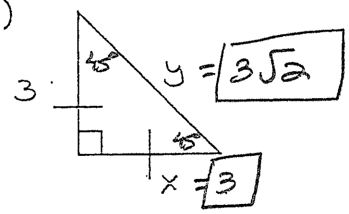
$\boxed{\frac{4\sqrt{3}}{3}} = y$ (Short leg)

$\frac{4}{\sqrt{3}} = \frac{y\sqrt{3}}{\sqrt{3}}$

$y = \frac{4}{\sqrt{3} \cdot \sqrt{3}} \cdot \sqrt{3} = \boxed{\frac{4\sqrt{3}}{3}}$

$x = 2\left(\frac{4\sqrt{3}}{3}\right) = \boxed{\frac{8\sqrt{3}}{3}}$

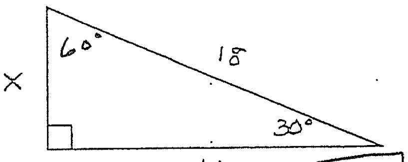
4)



$y = \boxed{3\sqrt{2}}$

$x = \boxed{3}$

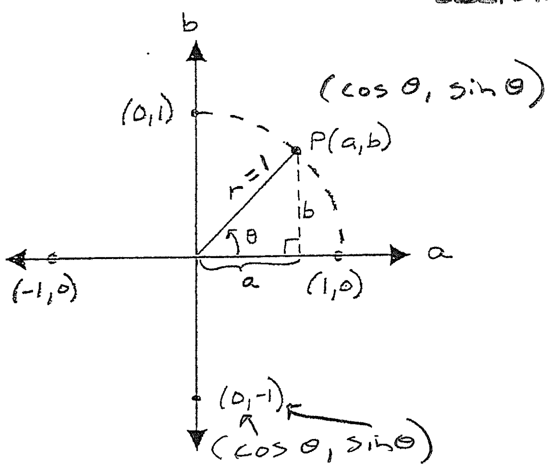
5)



$\boxed{9} = x$

$y = \boxed{9\sqrt{3}}$

Quadrantal Angles: angles whose terminal side lies on a coordinate axis



SOH CAH TOA

For convenience, we choose points 1 unit away from the origin. (Circle of radius 1)

$r=\sqrt{a^2+b^2}=1$ for each case.

$\sin\theta = \frac{b}{r} = \frac{b}{1} = b$ $\quad \csc\theta = \frac{r}{b} = \frac{1}{b}$

$\sin\theta = b$

$\cos\theta = \frac{a}{r} = \frac{a}{1} = a$ $\quad \sec\theta = \frac{r}{a} = \frac{1}{a}$

$\cos\theta = a$

$\tan\theta = \frac{b}{a} = \frac{\sin}{\cos}$ $\quad \cot\theta = \frac{a}{b} = \frac{\cos}{\sin}$

Example 1 - Evaluation Involving Quadrantal Angles

Find the following.

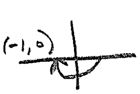
$\frac{3(180)}{2} = 270°$

a) $\sin(3\pi/2)$

$(0,-1)$
$(\cos, \sin)$

$= \boxed{-1}$

b) $\sec(-\pi)$ $\quad -180°$



$(-1,0)$

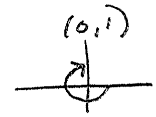
$\frac{1}{\cos} = \frac{1}{-1} = \boxed{-1}$

c) $\tan 90°$

$(0,1)$

$= \frac{\sin}{\cos} = \frac{1}{0} = \boxed{\text{und}}$

d) $\cot(-270°)$



$(0,1)$

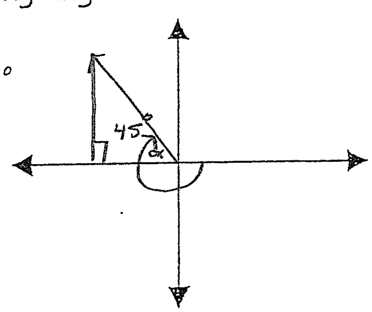
$\frac{1}{\tan} = \frac{1}{\frac{\sin}{\cos}} = \frac{\cos}{\sin} = \frac{0}{1}$
$= \boxed{0}$

Example 2 - Reference Triangles and Angles

Sketch the reference triangle and find the reference angle α for each of the following angles.

ACUTE + POSITIVE

a) $\theta = -225°$

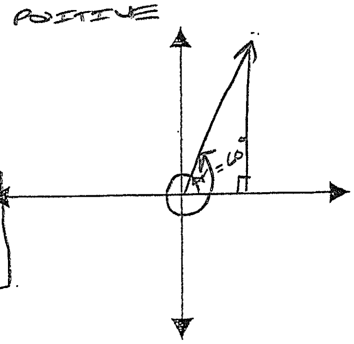


225
-180

$\boxed{\alpha = 45° \text{ reference angle}}$

b) $\theta = 420°$
-360

$\boxed{\alpha = 60° \text{ reference angle}}$

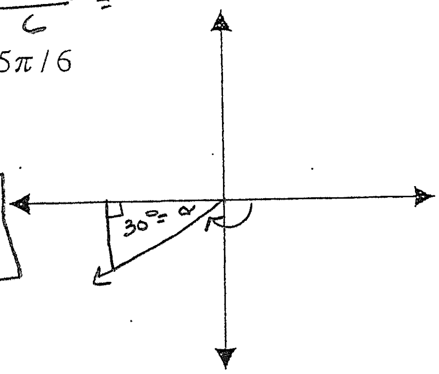


$\frac{-5(180)}{6} = -150°$

c) $\theta = -5\pi/6$

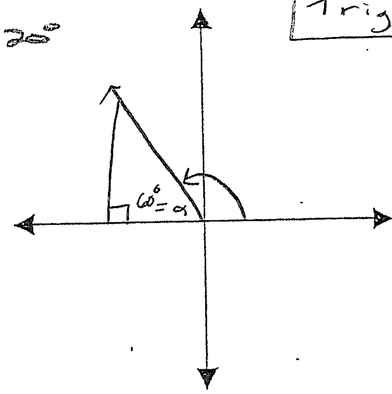
$180 - 150$

$\boxed{\alpha = 30 \text{ reference angle}}$



$\frac{2(180)}{3} = 120°$

d) $\theta = 2\pi/3$

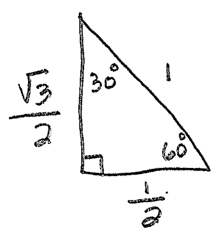


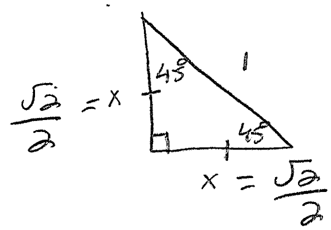
$180 - 120$

$\boxed{\alpha = 60° \text{ reference angle}}$

Example 3 - Exact evaluations for Special Angles and Real Numbers

Evaluate exactly.





$\frac{1}{\sqrt{2}} = \frac{x\sqrt{2}}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

A)

a) $\cos 60°$ $\frac{A}{H}$ $\quad \frac{\frac{1}{2}}{1} = \boxed{\frac{1}{2}}$

b) $\sin(\pi/3)$ $\quad \boxed{\frac{\sqrt{3}}{2}}$

$\frac{O}{H}$ $\quad \frac{180}{3} = 60°$

c) $\tan(\pi/3)$ $\quad \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \cdot \frac{2}{1} = \boxed{\sqrt{3}}$

$\frac{O}{A}$ $\quad \frac{180}{3} = 60°$

B)

a) $\sin 45°$ $\frac{O}{H}$ $\quad \boxed{\frac{\sqrt{2}}{2}}$

b) $\cot(\pi/4)$ $\quad \frac{1}{\tan} = \frac{1}{\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}} = \boxed{1}$

$\frac{180}{4} = 45°$

c) $\sec(\pi/4)$

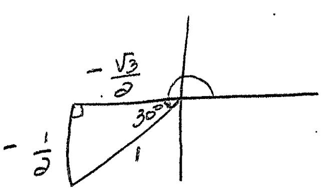
$\frac{1}{\cos} = \frac{1}{\frac{\frac{\sqrt{2}}{2}}{1}} = \frac{1}{\frac{\sqrt{2}}{2}} \cdot \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \boxed{\sqrt{2}}$

Example 4 - Exact Evaluation Continued

Evaluate exactly.

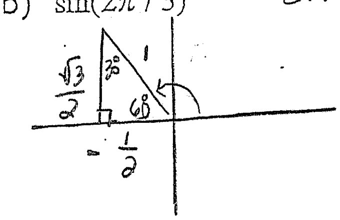
a) $\tan 210°$ $\quad \tan 30° = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} \cdot \frac{-2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$\boxed{\frac{\sqrt{3}}{3}}$



$\frac{2(180)}{3} = 120°$

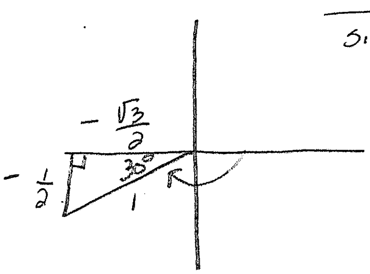
b) $\sin(2\pi/3)$ $\quad \sin 120° = \frac{\frac{\sqrt{3}}{2}}{1} = \boxed{\frac{\sqrt{3}}{2}}$



c) $\csc(-5\pi/6)$

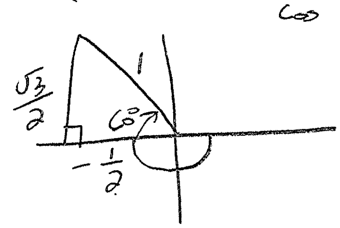
$\frac{-5(180)}{6} = -150°$

$\frac{1}{\sin} = \frac{1}{-\frac{1}{2}} ; -\frac{2}{1}$



$= \boxed{-2}$

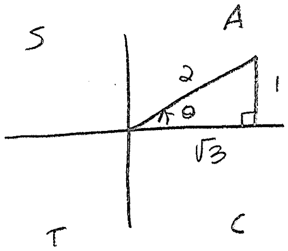
d) $\sec(-240°)$ $\quad \frac{1}{\cos} = \frac{1}{\frac{-\frac{1}{2}}{1}} = \boxed{-2}$



Example 5 – Finding Special Angles

Find the least positive θ in degrees and radian measure for which each is true.

a) $\tan\theta = 1/\sqrt{3}$ $\qquad \frac{O}{A} = \frac{1}{\sqrt{3}}$



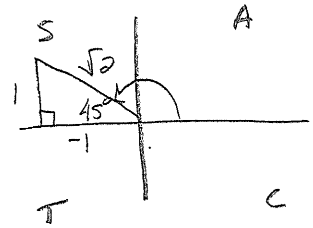
$\boxed{30^\circ \text{ or } \frac{\pi}{6}}$

$$\frac{30}{180} = \frac{x}{\pi}$$

$$= \frac{30\pi}{180} = \frac{180x}{180}$$

$$x = \frac{\pi}{6}$$

b) $\sec\theta = -\sqrt{2}$ $\qquad \cos = \frac{-1}{\sqrt{2}} \quad \frac{A}{H}$



$$\begin{array}{r} 180 \\ -45 \\ \hline 135^\circ \end{array}$$

$\boxed{135^\circ \text{ or } \frac{3\pi}{4}}$

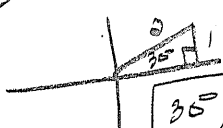
$$\frac{135}{180} = \frac{x}{\pi}$$

$$\frac{135\pi}{180} = \frac{180x}{180}$$

$$x = \frac{3\pi}{4}$$

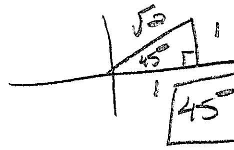
$\boxed{\text{p. 103 (51–56)}}$

(51) $\sin\theta = \frac{1}{2} \quad \frac{O}{H}$



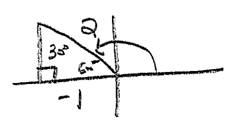
$\boxed{30^\circ, \frac{\pi}{6}}$

(52) $\cos\theta = \frac{1}{\sqrt{2}} \quad \frac{A}{H}$



$\boxed{45^\circ, \frac{\pi}{4}}$

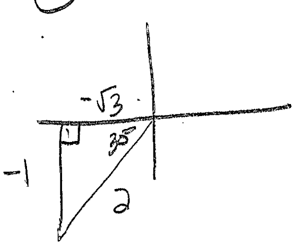
(53) $\cos\theta = -\frac{1}{2} \quad \frac{A}{H}$



$$\begin{array}{r} 180 \\ -60 \end{array}$$

$\boxed{120^\circ, \frac{2\pi}{3}}$

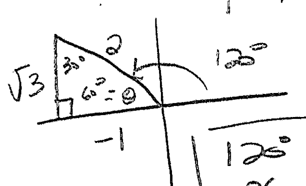
(54) $\sin\theta = -\frac{1}{2} \quad \frac{O}{H}$



$$\begin{array}{r} 180 \\ +30 \end{array}$$

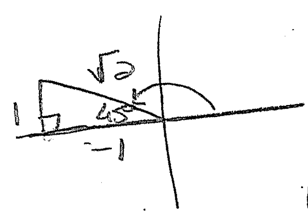
$\boxed{210^\circ \text{ or } \frac{7\pi}{6}}$

(55) $\tan\theta = -\frac{\sqrt{3}}{1} \quad \frac{O}{A}$



$\boxed{120^\circ \text{ or } \frac{2\pi}{3}}$

(56) $\cot\theta = -1 \qquad \frac{A}{O} = -1$



$\boxed{135^\circ \text{ or } \frac{3\pi}{4}}$